

# Colouring exact distance graphs of chordal graphs

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## Abstract

For a graph  $G = (V, E)$  and positive integer  $p$ , the *exact distance- $p$  graph*  $G^{[p]}$  is the graph with vertex set  $V$  and with an edge between vertices  $x$  and  $y$  if and only if  $x$  and  $y$  have distance  $p$ . Recently, there has been an effort to obtain bounds on the chromatic number  $\chi(G^{[p]})$  of exact distance- $p$  graphs for  $G$  from certain classes of graphs. In particular, if a graph  $G$  has tree-width  $t$ , it has been shown that  $\chi(G^{[p]}) \in \mathcal{O}(p^{t-1})$  for odd  $p$ , and  $\chi(G^{[p]}) \in \mathcal{O}(p^t \cdot \Delta(G))$  for even  $p$ . We show that if  $G$  is chordal and has tree-width  $t$ , then  $\chi(G^{[p]}) \in \mathcal{O}(p)$  for odd  $p$ , and  $\chi(G^{[p]}) \in \mathcal{O}(p \cdot \Delta(G))$  for even  $p$ .

Key Words: *exact distance graphs, chordal graphs, tree-width, bounded genus, adjacent-cliques graphs*

## 1 Introduction

All graphs in this paper are assumed to be finite, undirected, simple and without loops. For a graph  $G = (V, E)$  and vertices  $u, v \in V$ , we denote by  $d_G(u, v)$  (or  $d(u, v)$  when there is no danger of ambiguity) the distance between  $u$  and  $v$ , that is, the number of edges in a shortest path between  $u$  and  $v$ .

For a positive integer  $p$ , the  $p$ -th power graph  $G^p = (V, E^p)$  of  $G$  has the same vertex set as  $G$ , and the pair  $uv$  belongs to  $E^p$  if and only if  $d_G(u, v) \leq p$ .

Problems related to the chromatic number of graph powers were first considered by Kramer and Kramer [9, 10] in 1969 and have enjoyed significant attention ever since. It is clear that for  $p \geq 2$  any power of a star is a clique. Hence, in order to obtain bounds on  $\chi(G^p)$  we need to use the maximum degree  $\Delta(G)$  of  $G$ . One can easily see that any graph  $G$  with  $\Delta(G) \geq 3$  satisfies

$$\chi(G^p) \leq 1 + \Delta(G^p) \leq 1 + \Delta(G) \cdot \sum_{i=0}^{p-1} (\Delta(G) - 1)^i \in \mathcal{O}(\Delta(G)^p).$$

However, there are many classes of graphs for which much better bounds can be obtained. Recall that a graph is  $k$ -degenerate if every subgraph of  $G$  contains a vertex of degree at most  $k$ . Parametrising in terms of the degeneracy, Agnarsson and Halldórsson [1] gave upper bounds for many classes of graphs.

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**Theorem 1.1** (Agnarsson and Halldórsson [1]).

Let  $k$  and  $p$  be positive integers. There exists a constant  $c = c(k, p)$  such that for every  $k$ -degenerate graph  $G$  we have  $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$ .

Note that the exponent on  $\Delta(G)$  in this result is best possible, even for the class of trees, as the  $\Delta$ -regular tree with radius  $\lfloor p/2 \rfloor$  attests.

For some classes of graphs it is possible to obtain similar bounds without parametrising in terms of the degeneracy. Recall that a graph  $G$  is chordal if every cycle of  $G$  has a chord, i.e., if every induced cycle is a triangle. In [8], Král' proved that every chordal graph  $G$  with maximum degree  $\Delta$  satisfies  $\chi(G^p) \in \mathcal{O}(\sqrt{p}\Delta^{(p+1)/2})$  for even  $p$ , and  $\chi(G^p) \in \mathcal{O}(\Delta^{(p+1)/2})$  for odd  $p$ . Král' also showed that this upper bound for odd  $p$  is tight. It is worth mentioning that, in order to obtain this tight upper bound, Král' gave a simple proof of the already known fact that odd powers of chordal graphs are also chordal [2, 5].

Given that graphs with tree-width at most  $t$  have degeneracy at most  $t$ , Theorem 1.1 gives us an upper bound on  $\chi(G^p)$  when  $G$  belongs to a graph class with bounded tree-width. Although tree-width is usually defined in terms of tree-decompositions, an equivalent definition can be given in terms of chordal graphs, as follows.

**Definition 1.2.** The tree-width  $\text{tw}(G)$  of a graph  $G$  is the smallest integer  $t$  such that  $G$  is a subgraph of a chordal graph with clique number  $t + 1$ .

A notion related to graph powers is that of exact distance graphs. For a positive integer  $p$ , the exact distance- $p$  graph  $G^{[p]} = (V, E^{[p]})$  of  $G$  has the same vertex set as  $G$ , and the pair  $uv$  belongs to  $E^{[p]}$  if and only if  $d_G(u, v) = p$ . Clearly,  $E^{[p]}$  is a subset of the edge set of  $G^p$ , which means that  $\chi(G^{[p]}) \leq \chi(G^p)$ . We immediately see that Theorem 1.1 also gives an upper bound for the chromatic number of  $G^{[p]}$  in terms of the degeneracy and the maximum degree of  $G$ . However, when considering exact distance graphs, this upper bound is far from best possible. This is attested, for instance, by the following recent result of Van den Heuvel *et al.* [7].

**Theorem 1.3** (Van den Heuvel, Kierstead and Quiroz [7]).

- (a) Let  $p$  be an odd integer. For every graph  $G$  with tree-width at most  $t$  we have  $\chi(G^{[p]}) \leq t \cdot \binom{p+t-1}{t} + 1 \in \mathcal{O}(p^{t-1})$ .
- (b) Let  $p$  be an even integer. For every graph  $G$  with tree-width at most  $t$  we have  $\chi(G^{[p]}) \leq \left(t \cdot \binom{p+t}{t} + 1\right) \cdot \Delta(G) \in \mathcal{O}(p^t \cdot \Delta(G))$ .

This result actually extends to all classes with bounded expansion, a notion introduced by Nešetřil and Ossona de Mendez [12] which generalises classes closed under taking topological minors.

**Theorem 1.4.**

Let  $\mathcal{K}$  be a class of graphs with bounded expansion.

- (a) Let  $p$  be an odd positive integer. Then there exists a constant  $N = N(\mathcal{K}, p)$  such that for every graph  $G \in \mathcal{K}$  we have  $\chi(G^{[p]}) \leq N$ .
- (b) Let  $p$  be an even positive integer. Then there exists a constant  $N' = N'(\mathcal{K}, p)$  such that for every graph  $G \in \mathcal{K}$  we have  $\chi(G^{[p]}) \leq N' \cdot \Delta(G)$ .

Part (a) of Theorem 1.4 was first proved by Nešetřil and Ossona de Mendez [13]. Using the notion of generalised colouring numbers, Van den Heuvel *et al.* [7] proved Theorem 1.4 (b) and reproved Theorem 1.4 (a), obtaining much better bounds for  $N$ . For the case of classes with bounded tree-width they obtained the bounds in Theorem 1.3.

Our main result is a significant improvement on the bounds of Theorem 1.3 for chordal graphs. We prove this result in Section 4.

**Theorem 1.5.**

*Let  $G$  be a chordal graph with clique number  $t \geq 2$ .*

- (a) *For every odd integer  $p \geq 3$  we have  $\chi(G^{[\lfloor p \rfloor]}) \leq \binom{t}{2} \cdot (p+1)$ .*
- (b) *For every even integer  $p \geq 2$  we have  $\chi(G^{[\lfloor p \rfloor]}) \leq \binom{t}{2} \cdot \Delta(G) \cdot (p+1)$ .*

Although Definition 1.2 tells us that every graph of tree-width  $t$  is a subgraph of a chordal graph with clique number  $t+1$ , Theorem 1.5 does not extend to all graphs with tree-width at most  $t$ . We shall say more about this at the end of this section. Before that, let us state the full generality of our results and mention yet another direction in which we improve on existing bounds.

For two graphs  $G = (V, E)$  and  $G' = (V, E')$  on the same vertex set, define  $G \cup G' = (V, E \cup E')$ . For a fixed positive integer  $p$ , Theorem 1.5 trivially gives  $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)^s$  for any subset  $\{p_1, p_2, \dots, p_s\}$  of  $\{1, 2, \dots, p\}$  with  $q$  even elements. (Notice that if we take  $\{p_1, p_2, \dots, p_s\} = \{1, 2, \dots, p\}$ , then we have  $G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]} = G^p$ .) Taking a subset of even integers turns out to be quite different from taking a subset of odd integers. For even  $p$ , we note that the  $\Delta$ -regular tree of radius  $\lfloor p/2 \rfloor$ ,  $T_{\Delta, p}$ , shows that  $\chi(T_{\Delta, p}^{[\lfloor p/2 \rfloor]} \cup T_{\Delta, p}^{[\lfloor p/2 \rfloor + 1]} \cup \dots \cup T_{\Delta, p}^{[\lfloor p \rfloor]}) \in \Omega(\Delta^{p/2})$ . Hence, the bound of Theorem 1.1 gives again the right exponent on  $\Delta(G)$ . In contrast, we see that for odd  $p$  we obtain an upper bound on  $\chi(G^{[\lfloor p \rfloor - 1]} \cup G^{[\lfloor p \rfloor]})$  which does not depend on  $\Delta(G)$ . However, these trivial upper bounds stop being linear in  $p$ , even if we simply consider  $\chi(G^{[\lfloor p-2 \rfloor]} \cup G^{[\lfloor p \rfloor]})$ .

We prove Theorem 1.5 by proving the following stronger result which gives upper bounds on the chromatic number of all these gradations between  $G^{[\lfloor p \rfloor]}$  and  $G^p$ . For instance, these upper bounds are linear in  $p$  if the subsets of  $\{1, 2, \dots, p\}$  considered have bounded size. For the case of chordal graphs, these results greatly improve on the bounds of Theorem 1.3. Moreover, they improve on the constant term of Theorem 1.1.

**Theorem 1.6.**

*Let  $G$  be a chordal graph with clique number  $t \geq 2$ . Let  $p$  be a positive integer,  $S = \{p_1, p_2, \dots, p_s\} \subseteq \{1, 2, \dots, p\}$  and  $q$  be the number of even integers in  $S$ .*

- (a) *If  $1 \notin S$ , then we have  $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)$ .*
- (b) *If  $1 \in S$ , then we have  $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) \leq t \cdot \binom{t}{2}^{s-1} \cdot \Delta(G)^q \cdot (p+1)$ .*

Of course, if  $S = \{1\}$  then we have that  $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) = \chi(G) = t$ , given the well known fact that chordal graphs are perfect and hence satisfy  $\chi(G) = \omega(G)$ .

We obtain Theorem 1.6 by partitioning the graph  $G$  into levels. We fix a vertex  $x \in V(G)$  and we define the level  $\ell$  as the set of vertices having distance  $\ell$  with  $x$ . We bound the number

of colours needed to colour one level and then give different colours to levels which are at distance at most  $p$ . Apart from being natural in the context of exact distance graphs, this simple levelling argument is regularly used in colouring problems related to perfect graphs. (The real problem is, of course, in the analysis of each level.) Kündgen and Pelsmayer [11] used level partitions of chordal graphs to find an upper bound on the number of colours needed in a nonrepetitive colouring of a graph with tree-width  $t$ . More recently, Scott and Seymour [15] used level partitions to prove a conjecture of Gyárfás stating that there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every graph  $G$  with no odd hole.

Together with level partitions, the notion of adjacent-clique graphs is fundamental in our proof of Theorem 1.6. For a graph  $G$  and two cliques  $K$  and  $K^*$  in  $G$ , we say that  $K$  and  $K^*$  are *adjacent* if they are disjoint and there is a pair of vertices  $x \in K$ ,  $y \in K^*$  with  $xy \in E(G)$ . The *adjacent-cliques graph*  $AC(G)$  of a graph  $G$  has a vertex for each clique of  $G$ , and two vertices  $z$  and  $z^*$  of  $AC(G)$  are adjacent if and only if their corresponding cliques in  $G$  are adjacent. We prove the following result for the chromatic number of  $AC(G)$  when  $G$  is chordal.

**Theorem 1.7.**

*Let  $G$  be a chordal graph with clique number at most  $t$ . We have  $\chi(AC(G)) \leq \binom{t+1}{2}$ .*

We denote the line graph of a graph  $G$  by  $L(G)$ . It is easy to see that  $AC(G)$  contains  $G$  and  $L(G)^{[2]}$  as subgraphs. Hence, Theorem 1.7 tells us that for all chordal graphs  $G$  with clique number  $t$  there is an upper bound on  $\chi(L(G)^{[2]})$ . This is surprising, considering that, even for  $t = 2$ ,  $L(G)^{[p]}$  can have arbitrarily large cliques if  $p$  is odd (consider stars and subdivided stars). Whether or not there are upper bounds on  $\chi(L(G)^{[p]})$  for even  $p \geq 4$  is an open problem.

As we mentioned before, although every graph of tree-width  $t$  is a subgraph of a chordal graph of clique number  $t + 1$ , Theorem 1.5 and Theorem 1.6 do not extend to all graphs of tree-width at most  $t$ . This is because, if  $p \geq 2$ , it is possible for a subgraph  $H$  of a graph  $G$  to satisfy  $\chi(H^{[p]}) > \chi(G^{[p]})$ . However, we prove that for a class with bounded genus, an upper bound on  $\chi(G^{[p]})$  for all maximal graphs  $G$  in the class implies an upper bound on  $\chi(H^{[p]})$  for every graph  $H$  in the class, maximal or not.

**Proposition 1.8.**

*Let  $H$  be a graph with genus  $g \geq 0$ . There is an edge-maximal graph  $G$  of genus  $g$  such that  $H$  is a subgraph of  $G$  and  $\chi(H^{[p]}) \leq \chi(G^{[p]})$  for every positive integer  $p$ .*

The rest of the paper is organised as follows. In the next section we study the properties of level partitions of chordal graphs which will be essential for the proof of Theorem 1.6. In Section 3 we prove Theorem 1.7, and in Section 4 we complete the proof of Theorem 1.6. In Section 5 we prove Proposition 1.8. We conclude with a section on open problems and possible directions for further study.

## 2 Level partitions of chordal graphs

Let  $G$  be a graph and  $x$  be a fixed vertex of  $G$ . For any positive integer  $\ell$ , set  $N^\ell(x) = \{v \in V(G) \mid d(v, x) = \ell\}$ . We call  $N^\ell(x)$  the  $\ell$ -th level of  $G$  with respect to  $x$  and if we set  $N^0(x) = \{x\}$  we get that these levels partition the connected component of  $G$  containing  $x$ . We also set  $N^{<\ell}(x) = \bigcup_{i < \ell} N^i(x)$  and  $N^{>\ell}(x) = \bigcup_{i > \ell} N^i(x)$ .

Let  $G_k$ ,  $G_{<k}$  and  $G_{>k}$  be the graphs induced by  $N^k(x)$ ,  $N^{<k}(x)$  and  $N^{>k}(x)$ , respectively. Define the  $\ell$ -shadow of a subgraph  $H$  of  $G$  as the set of vertices in  $N^\ell(x)$  which have a neighbour in  $V(H)$ . We say that  $G$  is *shadow complete* (with respect to  $x$ ) if for every non-negative integer  $\ell$ , the  $\ell$ -shadow of every connected component of  $G_{>\ell}$  induces a complete graph.

Using a well-known theorem of Dirac [4] which characterises chordal graphs in terms of their minimal vertex cut sets, Kündgen and Pelsmayer [11] proved that connected chordal graphs are shadow complete with respect to any vertex.

**Lemma 2.1** (Kündgen and Pelsmayer [11]).

*Let  $G$  be a connected chordal graph with clique number  $t \geq 2$  and let  $x$  be any vertex in  $V(G)$ . Then  $G$  is shadow complete with respect to  $x$  and every  $G_\ell$  is a chordal graph with clique number strictly smaller than  $t$ .*

Before we start to see some implications of this lemma, let us state one additional definition. We say that a vertex  $v \in N^\ell(x)$ , is an *ancestor* (with respect to  $x$ ) of a vertex  $u \in N^m(x)$ ,  $\ell < m$ , if there is a path between  $u$  and  $v$  of length  $m - \ell$ . Clearly this path has exactly one vertex in each level  $N^\ell(x), N^{\ell+1}(x), \dots, N^m(x)$ .

The following result follows directly from Lemma 2.1.

**Corollary 2.2.**

*Let  $G$  be a connected chordal graph with clique number  $t \geq 2$ ,  $x \in V(G)$ , and  $u, v \in N^\ell(x)$  for some positive integer  $\ell$ . If  $u$  and  $v$  are both ancestors of some  $y \in N^{>\ell}(x)$ , then  $u$  and  $v$  are neighbours.*

With a bit more care we can prove that if two vertices are at the same level  $\ell$  and are at distance  $p$ , then their ancestors at level  $\ell - \lfloor p/2 \rfloor$  form cliques which either intersect or are adjacent.

**Lemma 2.3.**

*Let  $G$  be a connected chordal graph with clique number  $t \geq 2$ ,  $x \in V(G)$ , and  $u, v \in N^\ell(x)$  for some positive integer  $\ell$ . Suppose  $d(u, v) = p \geq 2$  and let  $K_u, K_v$  be the set of ancestors of  $u$  and  $v$  in  $N^{\ell - \lfloor p/2 \rfloor}(x)$ , respectively. We have that*

- (a) *if  $p$  is odd, then  $K_u$  and  $K_v$  are adjacent;*
- (b) *if  $p$  is even, then  $K_u$  and  $K_v$  are adjacent or  $K_u \cap K_v \neq \emptyset$ .*

*Proof.* Let  $k = \lfloor p/2 \rfloor$ , and note that we must have  $\ell \geq k$  as otherwise there would be a walk from  $u$  to  $v$  that goes through  $x$  and has length  $2\ell < 2k \leq p$ , which would contradict  $d(u, v) = p$ . We will prove (a) and (b) simultaneously by considering two possibilities for  $u$  and  $v$ .

We first consider the case in which  $u$  and  $v$  are in different components of  $G_{>\ell-k}$ . In this case it is clear that every path of length  $p$  joining  $u$  and  $v$  must contain a vertex from  $G_{\ell-k}$  (and no vertices in  $G_{<\ell-k}$ ). It is also easy to see that if  $p$  is even, then every path of length  $p$  joining  $u$  and  $v$  must have exactly one vertex in  $G_{\ell-k}$ . Since  $d(u, v) = p$ , this means that  $K_u \cap K_v \neq \emptyset$ . If  $p$  is odd, then every path of length  $p$  joining  $u$  and  $v$  must have exactly two vertices in  $G_{\ell-k}$ . This implies that  $K_u$  and  $K_v$  are adjacent.

We are now left to consider the case in which  $u$  and  $v$  are in the same connected component  $C$  of  $G_{>\ell-k}$ . Let  $z \in G_{\ell-k+1}$  be an ancestor of  $u$  and let  $z' \in G_{\ell-k+1}$  be an ancestor of  $v$ . Clearly,  $z$  and  $z'$  belong to  $C$ . We know by Lemma 2.1 that since  $G$  is chordal it is shadow complete, and so the neighbours of  $z$  and  $z'$  in  $G_{\ell-k}$  form a clique. This means that either  $K_u$  and  $K_v$  are adjacent or  $K_u \cap K_v \neq \emptyset$ . However, if  $p$  is odd we cannot have  $K_u \cap K_v \neq \emptyset$ .  $\square$

### 3 Adjacent-cliques graphs

In this section we prove Theorem 1.7. In order to prove this result we need to recall a specific characterisation of chordal graphs.

A *perfect elimination ordering* of a graph  $G$  is a linear ordering  $L$  of  $V(G)$  such that, for every vertex  $v \in V(G)$ , the neighbours of  $v$  which are smaller than  $v$  in  $L$  form a clique. The following classical result is proved in [6, Section 7].

**Proposition 3.1** (Fulkerson and Gross [6]).

*A graph is chordal if and only if it has a perfect elimination ordering.*

*Proof of Theorem 1.7.* By Proposition 3.1 we know  $G$  has a perfect elimination ordering. We fix one such ordering  $L$ . We say a vertex  $u$  is a *predecessor* of a vertex  $v$  if  $uv \in E(G)$  and  $u <_L v$ . Moving along the ordering  $L$ , we colour the vertices of  $G$  in the following way. A vertex  $v$  gets a colour  $a(v)$  which is different from  $a(u)$  if  $u$  is a predecessor of  $v$  or  $u$  is a predecessor of a predecessor of  $v$ . Since the clique number of  $G$  is at most  $t$  and since  $L$  is a perfect elimination ordering, each vertex has at most  $t - 1$  predecessors. Moreover, by choice of  $L$  we have that if  $v$  has  $r \leq t - 1$  predecessors, the largest (with respect to  $L$ ) of its predecessor has at most  $t - r$  predecessors which are not already predecessors of  $v$ ; the second largest predecessor of  $v$  has at most  $t - (r - 1)$  predecessors which are not already predecessors of  $v$ , and so on. Therefore, for any vertex  $v$  the set of predecessors and predecessors of a predecessor of  $v$  has size at most  $r + (t - r) + (t - (r - 1)) + \dots + (t - 1) \leq (t - 1) + 1 + 2 + \dots + t - 1 = \binom{t+1}{2} - 1$ . And so, the colouring  $a$  uses at most  $\binom{t+1}{2}$  colours.

We define a colouring  $c$  on the vertices of  $AC(G)$  in the following way. For every vertex  $z$  in  $AC(G)$ , with corresponding clique  $K$  in  $G$ , we set  $\mu(K)$  as the smallest vertex of  $K$  with respect to  $L$ . Every vertex  $z$  is assigned the colour  $c(z) = a(\mu(K))$ . We claim that  $c$  is a proper colouring of  $AC(G)$ .

Let  $z$  and  $z^*$  be adjacent vertices in  $AC(G)$ . We must prove that the corresponding cliques in  $G$ ,  $K$  and  $K^*$ , satisfy  $a(\mu(K)) \neq a(\mu(K^*))$ . Let  $u, u' \in K$  and  $v, v' \in K^*$  be vertices of  $G$  such that  $u = \mu(K)$ ,  $v = \mu(K^*)$  and  $u'v' \in E(G)$ . Without loss of generality we assume that  $u' <_L v'$ . If  $v = v'$ , we have that  $u'v \in E(G)$ . Otherwise, we have that both  $u'$  and  $v$  are predecessors of  $v'$ . Since  $L$  is a perfect elimination ordering, we also obtain  $u'v \in E(G)$ .



Note that  $a$  is a proper colouring of  $G$ . This means that if  $u = u'$ , we immediately get that  $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$  as desired. So assume  $u \neq u'$ . If  $v <_L u'$ , we have that both  $u$  and  $v$  are predecessors of  $u'$ , and so  $uv \in E(G)$ . This again gives us that  $a(\mu(K)) \neq a(\mu(K^*))$ . Otherwise, if  $u' <_L v$  we have that  $u'$  is a predecessor of  $v$ . Since  $u$  is a predecessor of  $u'$  we also obtain that  $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$  by definition of  $a$ .  $\square$

For a graph  $G$ , let  $AIC(G)$  be a graph with the same vertex set as  $AC(G)$  and an edge between two vertices  $z, z^*$  if and only if  $zz^* \in E(AC(G))$  or the corresponding cliques  $K$  and  $K^*$  have vertices in common. For later use, we note a property of the colouring  $c$  we constructed in the previous proof.

**Lemma 3.2.**

*Let  $G$  be a graph with clique number at most  $t$ . Colour the vertices of  $AIC(G)$  with the colouring  $c$ , defined in the proof of Theorem 1.7. If two vertices  $z, z^*$  of  $AIC(G)$  satisfy  $zz^* \in E(AIC(G))$  and  $c(z) = c(z^*)$ , then we have that the corresponding cliques  $K, K^*$  satisfy  $\mu(K) = \mu(K^*)$ .*

*Proof.* We prove that if  $\mu(K) \neq \mu(K^*)$ , then  $c(z) \neq c(z^*)$ . By the proof of Theorem 1.7 we know that if  $K$  and  $K^*$  have no vertices in common, then  $z$  and  $z^*$  get different colours. If  $K$  and  $K^*$  have vertices in common and  $\mu(K) \neq \mu(K^*)$ , we will prove that  $\mu(K)$  and  $\mu(K^*)$  are adjacent in  $G$ . Since  $a$  is a proper colouring of  $G$ , this will tell us that  $c(z) = a(\mu(K)) \neq a(\mu(K^*)) = c(z^*)$  which gives us the result.

If  $\mu(K)$  and  $\mu(K^*)$  are not adjacent in  $G$ , we have that neither of  $\mu(K), \mu(K^*)$  belong to  $K \cap K^*$ . Therefore, the minimum vertex  $v$  in  $K \cap K^*$  with respect to  $L$  is adjacent to  $\mu(K)$  and  $\mu(K^*)$ , and  $\mu(K), \mu(K^*)$  are smaller than  $v$  in  $L$ . But this contradicts the choice of  $L$ , since the neighbours of  $v$  which are smaller than  $v$  in  $L$  must form a clique, and so must be pairwise adjacent.  $\square$

## 4 Exact distance graphs of chordal graphs

Theorem 1.6 will follow from the next lemma.

**Lemma 4.1.**

*Let  $G$  be a connected chordal graph with clique number  $t \geq 2$ , let  $x$  be a vertex in  $G$ , and  $p \geq 2$  an integer. For any non-negative integer  $\ell$ , we have that*

- (a) *if  $p$  is odd, then there is a colouring  $h$  of  $N^\ell(x)$  using at most  $\binom{t}{2}$  colours such that if  $u, v \in N^\ell(x)$  satisfy  $uv \in E(G^{[\lfloor p \rfloor]})$ , then  $h(u) \neq h(v)$ ;*
- (b) *if  $p$  is even, then there is a colouring  $h'$  of  $N^\ell(x)$  using at most  $\binom{t}{2} \cdot \Delta(G)$  colours such that if  $u, v \in N^\ell(x)$  satisfy  $uv \in E(G^{[\lfloor p \rfloor]})$ , then  $h'(u) \neq h'(v)$ .*

*Proof.* Let  $k = \lfloor p/2 \rfloor$  and note, just as in the proof of Lemma 2.3, that if  $u, v \in N^\ell(x)$  satisfy  $uv \in E(G^{[\lfloor p \rfloor]})$ , then we must have  $\ell \geq k$ .

(a) By Lemma 2.1 we know that  $G_{\ell-k}$  is a chordal graph with clique number at most  $t-1$ . By Theorem 1.7 we know that there is a proper colouring  $c$  of the vertices of  $AC(G_{\ell-k})$  which uses at most  $\binom{t}{2}$  colours.

For every vertex  $y \in N^\ell(x)$  we consider the set of vertices  $K_y \subseteq N^{\ell-k}(x)$  which are ancestors of  $y$ . By Corollary 2.2 we have that  $K_y$  induces a clique in  $G_{\ell-k}$ . Let  $z_y$  be the vertex corresponding to this clique in  $AC(G_{\ell-k})$ . Define the colouring  $h$  by assigning  $h(y) = c(z_y)$  to every vertex  $y \in N^\ell(x)$ . Let  $u, v \in N^\ell(x)$  be such that  $uv \in E(G^{[p]})$ . By Lemma 2.3(a) we have that  $K_u$  and  $K_v$  are adjacent, which implies  $z_u z_v \in E(AC(G_{\ell-k}))$ . Therefore, we have  $h(u) = c(z_u) \neq c(z_v) = h(v)$ , as desired.

(b) Recall that in the proof of Theorem 1.7 the colouring  $c$  is obtained by ordering the vertices of the original graph according to a perfect elimination ordering. Also, for each clique  $K$  in the graph, we consider the minimum vertex  $\mu(K)$  in the clique with respect to the ordering.

We colour the vertices of  $AIC(G_{\ell-k})$  with the colouring  $c$ . Additionally, for each vertex  $w \in N^{\ell-k}(x)$  we choose an injective function  $b_w : N(w) \rightarrow \{1, \dots, \Delta(G)\}$ . For every vertex  $y \in N^\ell(x)$  we choose an arbitrary vertex  $\sigma(y)$  from  $N^{k-1}(y) \cap N(\mu(K_y))$ . The colouring  $h'$  assigns  $h'(y) = (c(z_y), b_{\mu(K_y)}(\sigma(y)))$  to every vertex  $y \in N^\ell(x)$ . Clearly  $h'$  uses at most  $\binom{t}{2} \cdot \Delta(G)$  colours.

Let  $u, v \in N^\ell(x)$  be such that  $uv \in E(G^{[p]})$ . We must show that  $h'(u) \neq h'(v)$ . Suppose we have  $K_u \cap K_v = \emptyset$ . By Lemma 2.3(b) we know that  $K_u$  and  $K_v$  are adjacent. As in part (a) we obtain  $c(z_u) \neq c(z_v)$  and so  $h'(u) \neq h'(v)$ , as desired.

If we have  $K_u \cap K_v \neq \emptyset$ , we know that  $z_u z_v \in E(AIC(G_{\ell-k}))$ . Let us assume  $c(z_u) = c(z_v)$ , as otherwise we would have  $h'(u) \neq h'(v)$ . By Lemma 3.2 we obtain that the corresponding cliques  $K_u$  and  $K_v$  satisfy  $\mu(K_u) = \mu(K_v)$ . Now notice that  $\sigma(u)$  must be different from  $\sigma(v)$ , as otherwise there would be a walk of length  $p-2$  joining  $u$  and  $v$  and going through  $\sigma(u)$ , which would contradict  $d(u, v) = p$ . Since  $b_{\mu(K_u)}$  is injective, we obtain  $h'(u) \neq h'(v)$ , as desired.  $\square$

*Proof of Theorem 1.6.* We may assume that  $G$  is connected. As we mentioned earlier, this theorem follows from Lemma 4.1. Here we prove (a) and leave (b) to the reader.

Fix a vertex  $x \in V(G)$ . Define a function  $f : V(G) \rightarrow \{0, \dots, p\}$  which satisfies  $f(u) = k$  for all  $u \in N^\ell(x)$  with  $\ell \equiv k \pmod{p+1}$ . For each level  $N^\ell(x)$  and integer  $p_i \in \{p_1, p_2, \dots, p_s\}$ , we define  $g_{\ell,i}$  as the colouring of  $N^\ell(x)$  guaranteed by Lemma 4.1, which assigns different colours to vertices of  $N^\ell(x)$  having distance  $p_i$ . To each vertex  $u \in N^\ell$  we assign a colour  $F(u) = (f(u), g_{\ell,1}(u), g_{\ell,2}(u), \dots, g_{\ell,s}(u))$ , and we do this for all  $\ell$ . Notice that for every  $1 \leq i \leq s$ , each vertex  $u \in N^\ell$  can only have distance  $p_i$  with vertices not in  $N^{<\ell-p}(x) \cup N^{>\ell+p}(x)$ . Hence, this colouring guarantees that, for all  $1 \leq i \leq s$ ,  $u$  gets a different colour from  $v$  whenever  $u$  and  $v$  have distance  $p_i$ .  $\square$

## 5 A remark on graphs with bounded genus

In this section we provide a way of obtaining from a connected graph  $H$  with genus  $g \geq 0$  another connected graph  $G$  of genus  $g$  which has the additional property of being edge-



maximal (with respect to having genus  $g$ ), and which satisfies  $\chi(H^{\lfloor p \rfloor}) \leq \chi(G^{\lfloor p \rfloor})$  for all positive  $p$ .

*Proof of Proposition 1.8.* We may assume  $V(H) \geq 3$ , as otherwise the result is trivial. We may also assume that  $H$  is connected.

Fix an embedding of  $H$  in a surface of genus  $g$ . We first construct from  $H$  a graph  $H'$  of genus  $g$  having the property that all of its faces have a cycle as its boundary. This can be done without altering distances by means of the following operation. Suppose  $y \in V(H)$  is a cut vertex. There is an ordering  $x_1, x_2, \dots, x_{|N(y)|}$  of the vertices in  $N(y)$  such that adding an edge between  $x_i$  and  $x_{i+1}$  (wherever such an edge does not already exist) would not create crossings. Using this ordering we add a path of length 2 between  $x_i$  and  $x_{i+1}$  (modulo  $|N(y)|$ ) if there is no edge joining the pair. Clearly  $y$  ceases to be a cut vertex after this operation, and no new cut vertices are created. We repeat this operation until there are no cut vertices. It is easy to see that  $H'$  satisfies that all of its faces have a cycle as its boundary, and that for every  $u, v \in V(H)$  we have  $d_H(u, v) = d_{H'}(u, v)$ .

If  $H'$  is not an edge-maximal graph of genus  $g$ , then there is a face of  $H$  having as its boundary a cycle  $C_k$ , with vertices  $z_0, \dots, z_{k-1}$ , for some  $k > 3$ . Inside this face we draw a cycle  $C_{k-1}$  with edges  $e_1, \dots, e_{k-1}$ . For all  $1 \leq i \leq k-1$ , we add edges joining  $z_i$  with the endvertices of  $e_i$ . We also add an edge joining  $z_0$  with the common endvertex of  $e_1$  and  $e_{k-1}$ . It is easy to see that this can be done in such a way that no crossings are made, the area between  $C_k$  and  $C_{k-1}$  is triangulated and  $C_{k-1}$  is the boundary of a face. (Figure 1 shows how to do this for  $k = 7$ .) We call the resulting embedded graph  $F$ . If  $F$  is not a triangulation, we repeat the operation on  $F$ , and we do this until we get a maximal graph  $G$  of genus  $g$ .

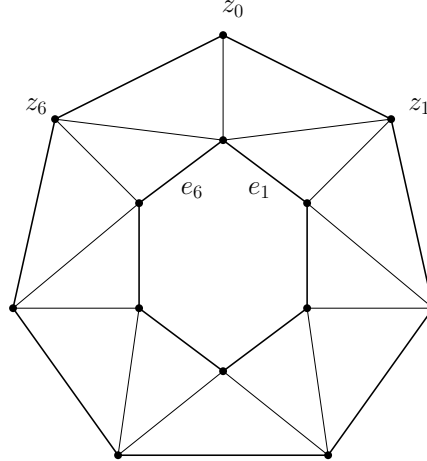


Figure 1: Drawing a  $C_6$  inside a face of  $H$  which has a  $C_7$  as its boundary.

To prove that  $\chi(G^{\lfloor p \rfloor}) \geq \chi(H^{\lfloor p \rfloor})$  for every positive integer  $p$ , it is enough to prove that  $F$  satisfies  $\chi(F^{\lfloor p \rfloor}) \geq \chi(H^{\lfloor p \rfloor})$  for every positive  $p$ . This can be done by showing that every pair of vertices  $u, v \in V(H)$  satisfies  $d_F(u, v) = d_H(u, v)$ . Note that if we had  $d_F(u, v) < d_H(u, v)$ , it would imply that there is a  $uv$ -path in  $F$  of length at most  $d_H(u, v) - 1$  containing vertices of  $C_{k-1}$ . In that case, we obtain that there is a pair of vertices  $x, y \in V(C_k)$  with  $d_F(x, y) <$

$d_H(x, y)$ , such that there is in  $F$  an  $xy$ -path of length at most  $d_H(x, y) - 1$  with vertices in  $C_{k-1}$  and no vertices outside of  $C_k \cup C_{k-1}$ . Hence, it is sufficient to show that every pair of vertices  $x, y \in V(C_k)$  satisfies  $d_{F[C_k \cup C_{k-1}]}(x, y) = d_{C_k}(x, y)$ . By the construction inside the face having  $C_k$  as its boundary, this is easy to check.  $\square$

## 6 Discussion and open problems

One of the main results of this paper, Theorem 1.5, gives upper bounds on the chromatic number of the exact distance graphs of chordal graphs of a given tree-width. These bounds greatly improve the existing ones because they are linear in the distance considered. Moreover, this dependency on the distance is very close to being best possible, as we shall describe below.

In [14] the following question from Van den Heuvel and Naserasr is mentioned: Is there a constant  $C$  such that for every odd integer  $p$  and every planar graph  $G$  we have  $\chi(G^{[p]}) \leq C$ ? Very recently Bousquet *et al.* [3] gave a negative answer to this question.

Recall that a graph is outerplanar if it can be embedded in the plane in such a way that all its vertices lie in the outer face. In order to settle the question above, Bousquet *et al.* constructed a family of chordal outerplanar graphs  $U_3, U_5, \dots$  with clique number 3 such that for every odd  $p \geq 3$  we have  $\chi(U_p^{[p]}) \in \Omega(\frac{p}{\log(p)})$ . On the other hand, by Theorem 1.5 (a) we know that  $\chi(U_p^{[p]}) \leq 3(p+1)$ . It would be interesting, then, to find the right growth rate in  $p$  for  $\chi(G^{[p]})$  when  $G$  is chordal and  $p$  is odd. It would also be interesting to find a tight bound for  $\chi(G^{[3]})$  when  $G$  is chordal and outerplanar. Van den Heuvel *et al.* [7] constructed a chordal outerplanar graph such that its exact distance-3 graph has chromatic number 5 (see Figure 2), whereas, Theorem 1.5 gives an upper bound of 12 for this class of graphs.

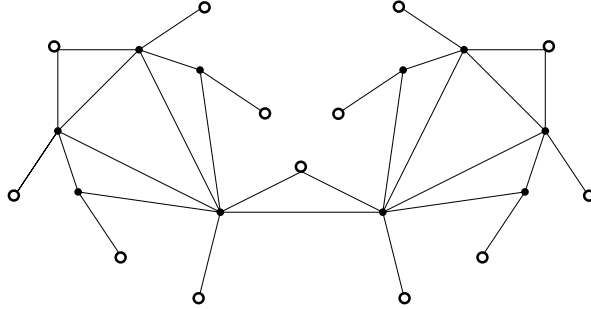


Figure 2: An outerplanar chordal graph  $G$  with  $\chi(G^{[3]}) = 5$ . Open vertices need at least 5 colours in a proper colouring of  $G^{[3]}$ .

A consequence of Theorem 1.6 (a) is that for every odd  $p$ , there is a constant  $N_{p,t}$  such that  $\chi(G^{[1]} \cup G^{[3]} \cup \dots \cup G^{[p]}) \leq N_{p,t}$  for all chordal graphs with clique number  $t$ . Nešetřil and Ossona de Mendez [14] gave a construction that shows that this constant must grow with  $p$ , even for outerplanar chordal graphs. Figure 3 gives a simpler construction with the same property. However, we note that the construction given by Nešetřil and Ossona de Mendez can be generalised to show that for every  $t \geq 3$  and odd positive  $p$ , there is a chordal graph  $G$  with clique number  $t$  such that  $\chi(G^{[1]} \cup G^{[3]} \cup \dots \cup G^{[p]}) \in \Omega(t^{\lfloor p/2 \rfloor + 1})$ . Meanwhile,

Theorem 1.6 (a) gives that  $\chi(G^{[1]} \cup G^{[3]} \cup \dots \cup G^{[p]}) \in \mathcal{O}(t^{2\lfloor p/2 \rfloor + 2})$ .



Figure 3: Outerplanar chordal graphs  $G$  for which  $\omega(G^{odd})$ , and hence  $\chi(G^{odd})$ , can be arbitrarily large.

For a graph  $G$ , a natural generalisation of  $G^{[1]} \cup G^{[3]} \cup \dots \cup G^{[p]}$  is the graph  $G^{odd}$ , which has the same vertex set as  $G$ , and  $xy$  is an edge in  $G^{odd}$  if and only if  $x$  and  $y$  have odd distance. Both of the constructions mentioned above tell us that even for outerplanar graphs  $G$  the chromatic number of  $G^{odd}$  can be arbitrarily large, because the clique number  $\omega(G^{odd})$  can be arbitrarily large. This inspired the following question of Thomassé, which appeared in [13] (see also [14]).

**Problem 6.1** ([13, Problem 11.2]).

*Is there a function  $f$  such that for every planar graph  $G$  we have  $\chi(G^{odd}) \leq f(\omega(G^{odd}))$ ?*

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